Observation: there are three kinds of salt, I, II, III.
We have the following informations:

	I	II	III
Price \$/kg	2	6	10
weight (kg)	4	8	12

Q: To mix all salt I, II, III together, what is the average of the price?

sol

~~Define f: Ω → {4, 8, 12}~~ ~~Let Ω = {I, II, III}~~

~~Define a function: $X : \Omega \to \{4, 8, 12\}$ {2, 6, 10}~~

~~$X(I) = 2$, $X(II) = 6$, $X(III) = 10$~~

Let $\Omega = \{I, II, III\}$

Put $X : \Omega \to \{2, 6, 10\}$

$X(I) = 2$, $X(II) = 6$, $X(III) = 10$

Note

the average of the price =

$$\frac{2\times 4 + 6\times 8 + 10\times 12}{4+8+12}$$

$$= 2\times\frac{4}{4+8+12} + 6\times\frac{8}{4+8+12} + 10\times\frac{12}{4+8+12}$$

$$= 2\times P\{X=2\} + 6\times P\{X=6\} + 10\times P\{X=10\}$$

From now on, all rvs are assumed to be discrete. 30

Def: Let X be a rv with $\operatorname{im} X = \{x_1, \dots, x_N\}$ $(1 \le N \le \infty)$. The expectation of X (or mean of X), (written $\bar{x}$, μ) write $E(X)$, is defined by

$$E(X) \overset{\circledast}{=} \sum_{i=1}^{N} x_i P\{X = x_i\} \quad \text{provided}$$

$\circledast$ is absolute convergent, i.e.

$$\sum_{i=1}^{N} |x_i| P\{X = x_i\} < \infty$$

Remark ① the absolute convergence of $\circledast$ assures that the series $\circledast$ does not depend on the rearrangement.

②: if $A \in \mathcal{E}$, then $E(I_A) = P(A)$.

Imp. ... we have

(i) $E(aX+b) = aE(X)+b$. a, b consts

$E(X+Y) = E(X) + E(Y)$

Prop: Let X, Y be discrete rvs. We have

(i) $E(aX+b) = aE(X)+b$. a, b consts

(ii) $E(X+Y) = E(X)+E(Y)$

(iii) $E(g \circ X) = \sum g(x_i) P\{X = x_i\}$, $g: \mathbb{R} \to \mathbb{R}$ is a Borel function

pf (i): Let $\operatorname{im} X = \{x_1, \dots, x_N\}$ (assume $x_i \neq x_j$)

Then $\operatorname{im}(aX+b) = \{ax_1+b, \dots, ax_N+b\}$

Hence $E(aX+b) = \sum (ax_i+b) P\{aX+b = ax_i+b\}$

$= \sum (ax_i+b) P\{X = x_i\}$

$= a\sum x_i P\{X=x_i\} + b\sum P\{X=x_i\}$

$= aE(X) + bP(\Omega) = aE(X)+b$

(ii) Let $\operatorname{im} Y = \{y_1, \dots, y_M\}$ $(y_i \neq y_j)$

~~Then~~ Note $\operatorname{im}(X+Y)$

$\operatorname{im}(X+Y) \subseteq \{x_i+y_j \mid i=1, \dots N, j=1, \dots M\}$

~~Hence~~ $\cancel{E(X+Y) = \sum_\omega X}$

Then $E(X+Y) = \sum_{i,j} (x_i+y_j) P\{X+Y = x_i+y_j\}$

$= \sum_{i,j} (x_i+y_j) P\{X=x_i, Y=y_j\}$

$= \sum_{i,j} x_i P\{X=x_i, Y=y_j\} + \sum_{i,j} y_j P\{X=x_i, Y=y_j\}$

$$= \sum_i x_i \sum_j P\{X=x_i, Y=y_j\} + \sum_j y_j \sum_i P\{X=x_i, Y=y_j\}$$

$$= \sum_i x_i P\{X=x_i\} + \sum_j y_j P\{Y=y_j\}$$

$\square$

Def: Let X be a discrete rv ~~Assume~~ The variance of X, write $\mathrm{var}(X)$, is defined by

$$\mathrm{var}(X) \equiv E[(X-\mu)^2] \quad \text{provided it exists}$$

where $\mu \equiv E(X)$.

Prop: (i) ~~var~~ $\mathrm{var}(X) = E(X^2) - E(X)^2$

(ii) $\mathrm{var}(aX+b) = a^2\,\mathrm{var}(X)$

(iii) ~~if X ... X are indep~~

if X, Y are indep rvs, then

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$$

pf (i). $\mathrm{var}(X) = E[$

pf (i). $\operatorname{var}(X) \equiv E[(X-\mu)^2]$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - \mu^2.$$

(ii). $\operatorname{var}(aX+b) = E[((aX+b)-\mu)^2]$

Note: $\mu_a \equiv E(aX+b) = a\mu + b$

Hence $\operatorname{Var}(aX+b) = E(aX+b - a\mu - b)^2$

$$= \cancel{E(a^2 X^2)} = E(a(X-\mu))^2$$

$$= a^2 E(X-\mu)^2$$

$\square$

From now on, let $\mathcal{B}(\mathbb{R})$ be the σ-algebra on $\mathbb{R}$ generated by $\{(-\infty, a] \mid a \in \mathbb{R}\}$

In this case, each event $A \in \mathcal{B}(\mathbb{R})$ is called a Borel subset of $\mathbb{R}$

Prop: A function $X: \Omega \to \mathbb{R}$ is a r.v iff $\{X \in A\} \in \xi$. $\forall A \in \mathcal{B}(\mathbb{R})$.

$\square$

$E(g \circ X) = \sum g(x_i) P\{X = x_i\}$

pf: let $\mathrm{im}\, g \circ X = \{y_1, \dots, y_M\}$, $\mathrm{im}\, X = \{x_1, \dots, x_N\}$

Then for each y_j, $\exists x_i \in X$ s.t. $g(x_i) = y_j$

Note $E(g \circ X) = \sum_j y_j P\{g \circ X = y_j\}$

$= \sum_j \sum_{i : g(x_i) = y_j} g(x_i) P\{X = x_i\}$ $\Big(\because \{\omega \mid g \circ X(\omega) = y_j\}$
$= \{\omega \mid g(x_i) = y_j, X(\omega) = x_i\}$
$= \coprod_{i : g(x_i) = y_j} \{X = x_i\} \Big)$

$= \sum_i g(x_i) P\{X = x_i\}$

~~Propn~~ $\square$

~~Propn Let X, Y be rvs. Then~~

~~$var(X+Y) = var$~~

~~Def: Let X, Y be rvs. The covariance of between X and Y is defined by~~

~~$cov(X, Y) = E(X - \mu_X)(Y - \mu_Y)$~~

Def: Let $X_1, \dots, X_N$ $(1 \leq N \leq \infty)$ be a sequence of rvs. We say that $(X_1, \dots, X_N)$ are independent if for any finitely many subsequence $X_{i_1}, \dots, X_{i_m}$, and ~~for~~ for any real numbers $a_1, \dots, a_m$,

$\{X_{i_1} \leq a_1\}, \dots, \{X_{i_m} \leq a_m\}$ are indep events.

Remark: $X_1, \dots, X_N$ are indep iff

$\{X_{i_1} \in A_1\}, \dots, \{X_{i_m} \in A_m\}$ are indep.

(for all Borel subsets $A_1, \dots, A_m$)

pendent iff

I_A, I_B are indep rvs

pf: "$\Rightarrow$" :

Case: $a < 0$, $b < 0$ $\Rightarrow$ $\{I_A \le a\} = \phi$, $\{I_B \le b\} = \phi$

$0 \le a < 1$, $b < 0$ $\Rightarrow$ $\{I_A \le a\} = A^c$, $\{I_B \le b\} = \phi$

$0 \le a < 1$, $0 \le b < 1$ $\Rightarrow$ $\{I_A \le a\} = A^c$, $\{I_B \le b\} = B^c$

$\vdots$

"$\Leftarrow$" Since $A^c = \{I_A \le \frac{1}{2}\}$ and $B^c = \{I_B \le \frac{1}{2}\}$

$\Rightarrow$ A^c, B^c are indep $\Rightarrow$ A, B are indep

Prop: Let X, Y be two indep rvs & let

$f, g : \mathbb{R} \to \mathbb{R}$ be two Borel functions.

Then $f \circ X$, $g \circ Y$ are indep rvs

and ~~$E(g\circ X)$~~ $E((f X)\, g(Y)) =$

$E(f(X))\, E(g(Y))$

pf: Note: $\{f \circ X \le a\} = \{X \in f^{-1}(-\infty, a)\}$

$\{g \circ Y \le b\} = \{Y \in g^{-1}(-\infty, b)\}$

$\therefore$ $f(X)$ and $g(Y)$ are indep.

Claim: $E(f(X)\, g(Y)) = E(f(X))\, E(g(Y))$

pf claim: Since $f(X)$, $g(Y)$ are indep, 36

need to show that if X, Y are indep, then

$E(XY) = E(X)E(Y)$

Let $Z \equiv XY$, and $\mathrm{im}\, Z = \{z_1, \ldots, z_\ell\}$

$z_k = x_i y_j$ for some x_i and y_j

Then $E(XY) = \sum_k z_k P\{Z = z_k\}$ $\left(\because \{Z = z_k\} = \bigsqcup_{i,j : x_i y_j = z_k} \{X = x_i, Y = y_j\} \right.$

$= \sum_k \sum_{i,j : x_i y_j = z_k} x_i y_j P\{X = x_i, Y = y_j\}$

$= \sum_{i,j} x_i y_j P\{X = x_i\} P\{Y = y_j\} = E(X)E(Y)$

$\square$

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Def: Let X, Y be rvs. The covariance of X and Y, write $\mathrm{cov}(X, Y)$, is defined by

$$\mathrm{cov}(X, Y) \equiv E[(X-\mu_X)(Y-\mu_Y)]$$

Prop: If X, Y are independent, then $\mathrm{cov}(X, Y) = 0$

Prop: Let $X_1, X_2 \ldots X_N$ $(1 \leq N < \infty)$ be rvs. Then

$$\mathrm{var}(X_1 + \cdots + X_N) = \sum_{i=1}^{N} \mathrm{var}(X_i) + \sum_{i \neq j} \mathrm{cov}(X_i, X_j)$$

In particular, if $X_1, \ldots X_N$ are indep, then

$$\mathrm{var}(X_1 + \cdots + X_N) = \mathrm{var}(X_1) + \cdots + \mathrm{var}(X_N)$$

Pf: $\mathrm{var}(X_1 + \cdots + X_N) = E[(X_1 + \cdots + X_N - \mu_{X_1 + \cdots + X_N})^2]$

$$= E\left(\left[\sum_{i=1}^{N} (X_i - \mu_{X_i})\right]^2\right)$$

$$= \sum_{i=1}^{N} E(X_i - \mu_{X_i})^2 + \sum_{i \neq j} E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})]$$

$\square$

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lemma: Let $a_1 < \dots < a_n$ and $(p_1, \dots, p_n) \in \mathbb{R}^n$

with $0 \le p_i \le 1$, $\sum_{i=1}^{n} p_i = 1$

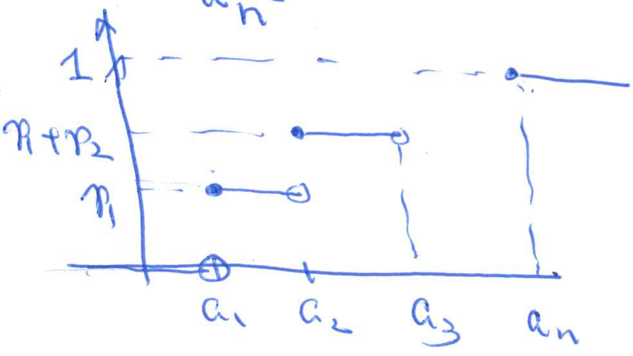
Let $f : \{a_1, \dots, a_n\} \to \{p_1, \dots, p_n\}$ be a function such that $f(a_i) = p_i$.

Then there is a prob space (Ω, ξ, P) and a rv $X : \Omega \to \{a_1, \dots, a_n\}$ such that

$P\{X = a_i\} = p_i$ for all $i = 1, \dots, n$.

pf: Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \begin{cases} 0 & t < a_1 \\ p_1 + \dots + p_i & a_i \le t < a_{i+1} \\ 1 & a_n \le t \end{cases}$$



Then $\exists (\Omega, \xi, P)$ and a rv $X : \mathbb{R} \to \mathbb{R}$

st $F_X = F$ ie:

$$F_X(t) = F(t), \quad \forall t \in \mathbb{R}$$

Then $P\{X = a_i\} = \cancel{P\{X \le a_i\}} \lim_{n \to \infty} P\{a_i - \frac{1}{n} < X \le a_i\}$

$= \lim_{n \to \infty} [F_X(a_i) - F_X(a_i - \frac{1}{n})] = p_i$

Def: Assume that there are n independent trials and the probability of success in each trial is p. Let X ~~den the~~ denote the number of successes. Then X is called a binomial random variable with parameter n, p. Write $X \sim b(n,p)$

Remark: There is a prob sp (Ω, Σ, P) and a rv X st $X \sim b(n,p)$

Define: $\Omega = \{(x_1, \ldots, x_n) \mid x_i = 0 \text{ or } 1\}$

$$\Sigma = \mathcal{P}(\Omega)$$

Define $P\{(x_1, \ldots, x_n)\} = p^i (1-p)^{n-i}$

~~where i = the number of x_k st $x_k = 1$~~

and

$X : \Omega \to \{0, 1, \ldots, n\}$

$X(x_1, \ldots, x_n) = i$

where i = the number of x_k st $x_k = 1$.

Then $X \sim b(n,p)$

Prop: Let $X \sim b(n,p)$. Then

(i): $P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$

(ii) $E(X) = np$

(iii) $\mathrm{var}(X) = np(1-p)$.

pf: (ii). $E(X) = \sum_{k=0}^{n} k P\{X=k\}$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} p \cdot p^{k-1} (1-p)^{n-1-(k-1)}$$

$$= np(p+(1-p))^{n-1} = np.$$

(iii). $\mathrm{var}(X) = E(X^2) - E(X)^2$

$$E(X^2) = \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k\, n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{(k-1)\, n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} + \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} + \sum_{j=0}^{n-1} \frac{n!}{j!(n-1-j)!} p^{j+1} (1-p)^{n-1-j}$$

$(i=k-2)$ $(j=k-1)$

$$= \sum_{i=0}^{n-2} \frac{n!}{i!(n-2-i)!} p^{i+2} (1-p)^{n-2-i} + \quad ''$$

$= p^2 n(n-1)[p+(1-p)]^{n-2} + np(p+(1-p))^{n-1}$ ④

$\therefore \quad Var(X) = E(X^2) - E(X)^2$

$= n(n-1)p^2 + np - n^2p^2$

$= -np^2 + np = np(1-p)$ $\square$

Alternative proof:

Define $X_i : \Omega \longrightarrow \mathbb{R}$ by

$$X_i = \begin{cases} 1 & \text{if } i\text{-th trial success} \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = X_1 + \cdots + X_n$

Note: $E(X_i) = p$

$var(X_i) = E(X_i^2) - E(X_i)^2$

$= E(X_i) - E(X_i)^2$

$= p - p^2 \quad \forall i$

$\square$

Observation:

Let $X_n \sim b(n,p)$. and $\lambda = np$.

Fix λ. and $k \in \mathbb{N}$.

$$P\{X_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= 1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\cdot\frac{1}{k!}\lambda^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\longrightarrow \frac{1}{k!}\lambda^k e^{-\lambda}$$

Def: A discrete rv $X : \Omega \to \{0,1,2\ldots\}$ is called a Poisson rv with parameter $\lambda > 0$. if

$$P\{X = k\} = \frac{1}{k!} e^{-\lambda} \cdot \lambda^k, \quad \forall k = 0,1,2\ldots$$

Remark: Define : $F : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F(t) = \begin{cases} 0 & t < 0 \\ \cancel{\frac{1}{k!}e^{-\lambda}\lambda^k} & \cancel{k \le t < k+1} \\ \sum_{i=0}^{k} \frac{1}{i!} e^{-\lambda} \cdot \lambda^i & k \le t < k+1 \end{cases}$$

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Note F is right cont and

$$\lim_{t\to\infty} F(t) = \lim_{k\to\infty} \sum_{i=0}^{k} e^{-\lambda} \frac{\lambda^i}{i!} = 1$$

$\therefore$ Such a Poisson rv exists ~~for all~~. $\square$

Prop: If X is a Poisson rv with a parameter $\lambda > 0$, then $E(X) = \lambda = \mathrm{var}(X)$

pf: $\vdash_1$: $E(X) = \lambda$

Note: $E(X) = \sum_{k=0}^{\infty} k P\{X=k\} = \sum_{k=0}^{\infty} k \cdot \frac{1}{k!} \lambda^{k} e^{-\lambda}$

$$= \left(\sum_{k=1}^{\infty} \frac{1}{(k-1)!} \lambda^{k} \right) e^{-\lambda}$$

$$= \left(\sum_{j=0}^{\infty} \frac{1}{j!} \lambda^{j+1} \right) e^{-\lambda} = \lambda$$

$(j = k-1)$

$\vdash_2$: $\mathrm{var}(X) = \lambda$: Note $\mathrm{var}(X) = E(X^2) - E(X)^2$

Note: $E(X^2) = \sum_{k=0}^{\infty} k^2 P\{X=k\} = \sum_{k=1}^{\infty} k^2 e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$$= \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k (k-1)}{(k-2)!} + \left(\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \right) e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} + \left(\sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!} \right) e^{-\lambda}$$

$$= e^{-\lambda} \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \right) \lambda^2 + \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \right) \lambda e^{-\lambda} \quad j = k-1$$

$= \lambda^2 + \lambda$ $\quad \therefore \mathrm{var}(X) = \lambda$ $\square$

Ch 6. Continuous random variables

(14)

Fix a prob space (Ω, Σ, P).

Notation:

Let $\mathcal{L}^1$ denote the class of (Riemann integrable) functions: $f: \mathbb{R} \to \mathbb{R}$ which satisfies the condition:

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Def.: A rv $X: \Omega \to \mathbb{R}$ is called a continuous rv with a density function $f_X: \mathbb{R} \to [0, \infty)$ which satisfies the conditions: $f_X \in \mathcal{L}^1$ and

$$F_X(T) \equiv P\{X \le T\} = \int_{-\infty}^{T} f_X(t)\, dt \quad \text{, for all } T \in \mathbb{R}$$

Remark: (1) f_X is "almost" uniquely determined by X. In fact: $F_X'(T) = f_X(T)$ for "almost" all T.

(2): $$\int_{-\infty}^{\infty} f_X(t)\, dt = 1.$$

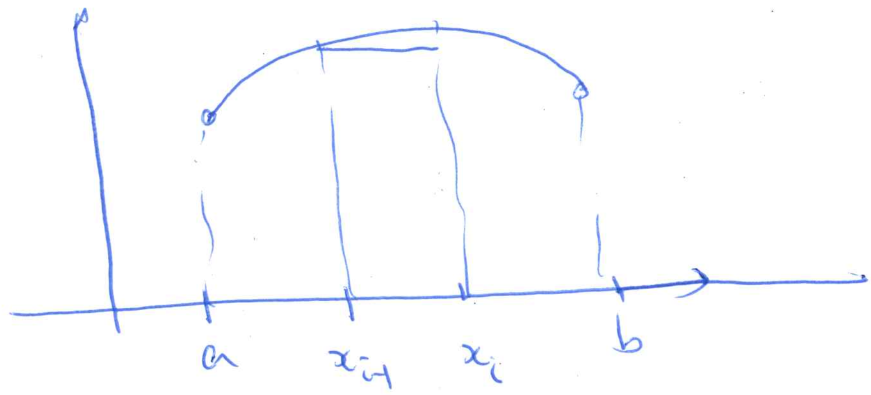
(3) $P\{X = a\} = 0$. for all $a \in \mathbb{R}$

(4): F_X is a cont function.

Observation:

Let $X : \Omega \to \mathbb{R}$ be a continuous rv with density function $f : \mathbb{R} \to [0, \infty)$

Assume that $f \equiv 0$ on $\mathbb{R} \setminus [a, b]$



Let $a = x_0 < x_1 < \dots < x_N = b$

Define: $X_N : \Omega \to \mathbb{R}$ by

$$X_N(\omega) = x_i \quad \text{if } x_{i-1} < X(\omega) \le x_i$$

Then $X_N : \Omega \to \{x_1, \dots, x_N\}$ is a discrete r.v.

Note: $E(X_N) = \sum_{i=1}^{N} x_i P\{X_N = x_i\}$

$$= \sum_{i=1}^{N} x_i \int_{x_{i-1}}^{x_i} f(t)\,dt$$

$$\approx \sum_{i=1}^{N} \tilde{x}_i f(\tilde{x}_i) \Delta x_i$$

Define: Let $X : \Omega \to \mathbb{R}$ be a continuous rv with density f. The expectation of X, $E(X)$, is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if $\int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$.

~~Prop: Let X, Y be cont rvs. Then~~

$$\cancel{E(X+Y) = E(X) + E(Y)}$$

eg: Let X be a cont rv with density

$$f(x) = \frac{c}{1+x^2}, \qquad x \in \mathbb{R}$$

Find c.

Sol: Since $\int_{-\infty}^{\infty} f(x)\, dx = 1$, $\quad c\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1$

$\Rightarrow \quad c \tan^{-1} x \Big|_{-\infty}^{\infty} = 1 \qquad \Rightarrow c\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$

$\Rightarrow \quad c = \frac{1}{\pi}$

(X is called a Cauchy rv)

Remark: $E(X)$ ~~is~~ does not exist. in this case

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Note: $$\int_{-\infty}^{\infty} |x| f(x)\,dx = \int_{-\infty}^{\infty} \frac{|x|\,dx}{\pi(1+x^2)}$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{x\,dx}{1+x^2} = \frac{1}{\pi}\int_0^{\infty} \frac{dx^2}{1+x^2} =$$

$$\frac{1}{\pi} \ln(1+x^2)\Big|_0^{\infty}$$ does not exist!

Prop: Let X be a cont rv with density f.
Let $g: \mathbb{R} \to \mathbb{R}$ be a Borel funct such that ~~$g \circ f(X)$~~ $E(g(X))$ exists. then

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

pf (Idea): Case: ~~Assume g is~~ Let $g = I_E$ for some $E \in \mathcal{B}(\mathbb{R})$

Note: $g \circ X(\omega) = I_E(X(\omega)) = I_{X^{-1}(E)}(\omega)$

$$E(g \circ X(\omega)) = P\{\omega \mid X(\omega) \in E\}$$
$$= \int_E f(x) dx = \int_{\mathbb{R}} I_E(x) f(x) dx$$

Case: $g = \sum_{i=1}^{N} a_i \chi_{E_i}$ simple function.

Case: $s_n \uparrow g$. ~~s_n is a~~ s_n is a simple function in $\mathbb{R}$

Def: Let X be a rv. The variance of X is defined ~~by~~ by $var(X) = E[(X-\mu)^2]$, where $\mu = E(X)$.

Prop: $var(X) = E(X^2) - E(X)^2$
Note: if X is cont, then

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx.$$

~~Def~~ e.g. A cnt random variable X is said to be uniformly ~~dst~~ distributed over (a,b) if it has density

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

Then $E(X) = \frac{b+a}{2}$ and $var(X) = \frac{(b-a)^2}{12}$.

pf:- $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b \frac{x}{b-a} dx$

$$= \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b = \frac{1}{2}(b+a)$$

$Var(X) = E(X^2) - E(X)^2$

Note: $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_a^b \frac{x^2}{b-a} dx$

$$= \frac{1}{3}(a^2 + ab + b^2)$$

$\Rightarrow var(X) = E(X^2) - E(X)^2 = \frac{(b-a)^2}{12}$

$\square$

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eg: Let X be a cont rv with density

$$f(x) = \begin{cases} 2e^{-2x} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = \sqrt{X}$.

Find f_Y and $E(Y)$

Sol: Note: $P\{X \le a\} = 0$. for all $a<0$

Hence X may ~~be~~ viewed as $X \ge 0$.

Aim to find $f_Y : \mathbb{R} \to [0,\infty)$ such that

$$P\{Y \le T\} = \int_{-\infty}^{T} f_Y(y)\,dy. \quad \text{for all } T$$

Note: $P\{Y \le T\} = P\{\sqrt{X} \le T\} = P\{X \le T^2\}$

$$= \int_{-\infty}^{T^2} f_X(x)\,dx = \int_0^{T^2} 2e^{-2x}\,dx$$

Let $y = \sqrt{x}$, $x \geq 0$

Then $dy = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}\,dx \Rightarrow 2y\,dy = dx$

Then $P\{Y \le T\} = \int_0^T 2e^{-2y^2} \cdot (2y)\,dy = 4ye^{-2y^2}$

Note $P\{Y \le 0\} = 0$

$$\therefore f_Y(y) = \begin{cases} 4y\,e^{-2y^2} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

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$$E(Y) = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \int_0^{\infty} y \cdot 4 y \, e^{-2y^2} \, dy.$$

Alternative:

$$E(Y) = E(\sqrt{X}) = \int_0^{\infty} \sqrt{x} \; 2e^{-2x} \, dx$$

$\square$

Def: A ~~r.v.~~ cont r.v. X ~~s~~ is said to ~~be~~ be a normal r.v. with parameter μ, σ^2 $(\sigma > 0)$ if it has density

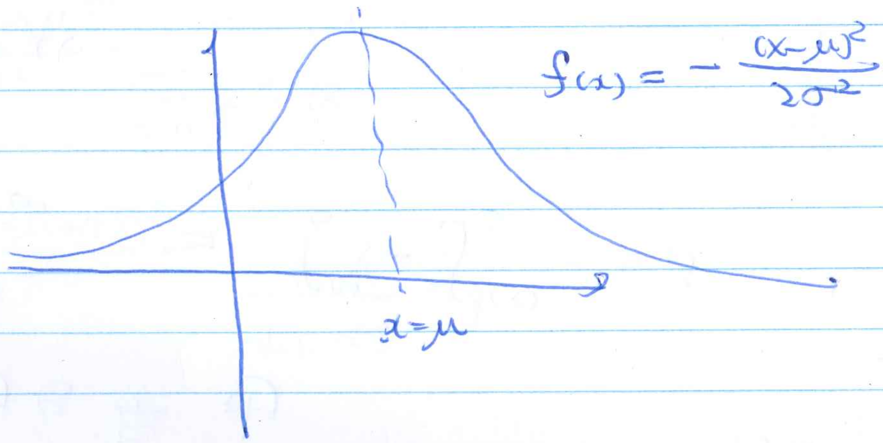
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in (-\infty, \infty)$$

In this case, write $X \sim N(\mu, \sigma^2)$

If $\mu = 0$, $\sigma = 1$, $X \sim N(0,1)$ is called a standard normal r.v.

Fact: • $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

•



• $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \overset{(*)}{=} 1$

Reason: Put $y = \frac{x-\mu}{\sigma}$

Then $dy = \frac{1}{\sigma} dx$. Since $\sigma > 0$,

$\Rightarrow (*) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2}} \sigma\, dy = \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1$

• $X \sim N(\mu, \sigma^2)$ exists !

Prop: If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$ and $var(X) = \sigma^2$.

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pf: $E(X) = \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} dx$

$= \int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} dx + \int \mu \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$\int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} dx,$

$y = x-\mu$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} y e^{\frac{-y^2}{2\sigma^2}} dy$

$= \int_0^{\infty} + \int_{-\infty}^{0} = \int_0^{\infty} - \int_0^{\infty} = 0$

$\therefore \quad E(X) = \mu$

$var(X) = E(X-\mu)^2$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$y = \frac{x-\mu}{\sigma}$

$= \int_{-\infty}^{\infty} \frac{\sigma^2}{\sqrt{2\pi}} y^2 e^{-\frac{y^2}{2}} dy$

$= \int_{-\infty}^{\infty} \frac{\sigma^2}{\sqrt{2\pi}} (-y) \, d e^{\frac{-y^2}{2}}$

$= \frac{\sigma^2}{\sigma\sqrt{2\pi}} (-y) e^{-\frac{y^2}{2}} \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$

$= \sigma^2$

$\square$

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Prop: If $X \sim N(\mu, \sigma^2)$, then $\alpha X + \beta \sim N(\alpha\mu + \beta, \alpha^2\sigma^2)$

where α, β consts and $\alpha > 0$

Hence if $Y = \dfrac{X-\mu}{\sigma}$, then $Y \sim N(0,1)$

pf: ~~N.B.~~

$$P\{\alpha X + \beta \le T\} = P\left\{X \le \frac{T-\beta}{\alpha}\right\}$$

$$= \int_{-\infty}^{\frac{T-\beta}{\alpha}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\,\sigma}\, dx$$

$$= \int_{0}^{T} e^{-\frac{\left(\frac{y-\beta}{\alpha}-\mu\right)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \frac{1}{\alpha}\, dy$$

$\left(y = \alpha x + \beta \right.$

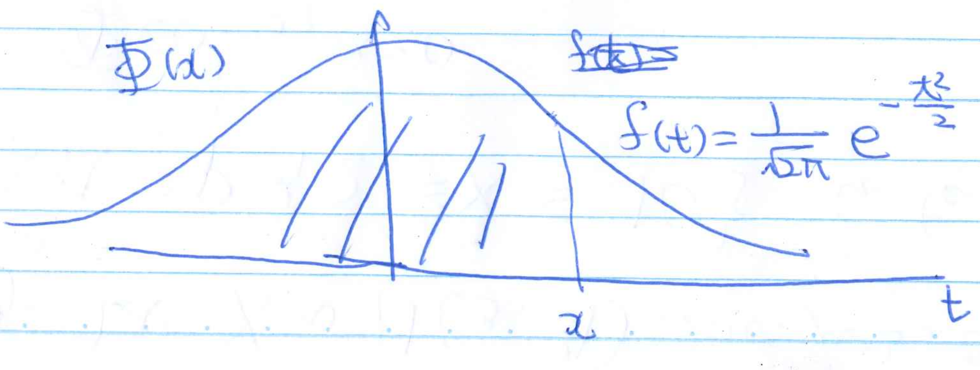
$\left(\Rightarrow dy = \alpha dx \right.$

$$= \int_{0}^{T} e^{\frac{[y-(\alpha\mu+\beta)]^2}{2\alpha^2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\,\alpha\sigma}\, dy$$

$\square$

Notation

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$



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eg: Let $X \sim N(3, 4)$, ie: $\mu = 3$, $\sigma = 2$

Find $P\{2 \le X \le 10\}$ in terms of

$\Phi(x)$, $x \ge 0$

sol: $P\left\{\frac{2-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{10-\mu}{\sigma}\right\}$

$= P\left\{\frac{2-3}{2} \le \frac{X-3}{2} \le \frac{10-3}{2}\right\}$

$= P\left\{-0.5 \le \frac{X-3}{2} \le 3.5\right\}$

$= \Phi(3.5) - \Phi(-0.5)$

$= \Phi(3.5) - (1 - \Phi(0.5))$

$= \Phi(3.5) + \Phi(0.5) - 1$

$\square$

H.W 4 : ~~[illegible]~~

~~[illegible]~~

p182, ~~[illegible]~~, 5.

p212, 10

p258, 4, 6

6. Nov 2014

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Th: (De Moivre-Laplace th): Let $X_n \sim b(n, p)$. Then
(1738)

$$\lim_{n\to\infty} P\left\{ a < \frac{X_n - np}{\sqrt{np(1-p)}} < b \right\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}}\, dt$$

$$= \Phi(b) - \Phi(a)$$

Remark: Recall: $\mu_{X_n} = np$

$$\sigma_{X_n} = \sqrt{np(1-p)}$$

Hence $P\left\{ \frac{X_n - \mu}{\sigma} \le T \right\} \longrightarrow P\{ Z \le T \}$

where $Z \sim N(0, 1)$

e.g.: If $X \sim b(100, 0.5)$, approximate 56

$$P\{X \geq 65\}$$

Note: $P\{X \geq 65\} = P\{X \geq 64.5\}$

$$\approx = P\left\{\frac{X-np}{\sqrt{np(1-p)}} \geq \frac{64.5-np}{\sqrt{np(1-p)}}\right\}$$

$n = 100$, $p = 0.5$

$$\approx P\left\{\frac{X-100\times0.5}{\sqrt{100\times0.5\times0.5}} \geq 2.9\right\}$$

$$= 1 - \Phi(2.9) \approx 0.0019$$

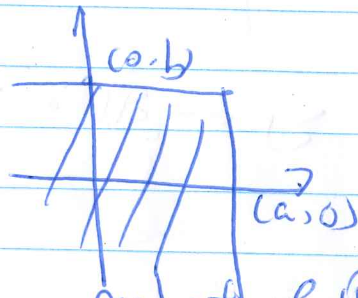
J

Ch 8+9: Joint Distributions

Throughout ~~ta~~ this chapter, let (Ω, ξ, P) be a prob space and let $X, Y: \Omega \to \mathbb{R}$ be rvs

Notation:

- ~~The~~ Write $\mathcal{B}(\mathbb{R}^2)$ for the σ-alg on $\mathbb{R}^2$ generated by $(-\infty, a]\times(-\infty, b]$, $(a,b)\in\mathbb{R}^2$



~~Aned~~ ~~Aned~~ And every element $E\in\mathcal{B}(\mathbb{R}^2)$ is called a Borel subset of $\mathbb{R}^2$.

Remark: All "regular" shapes of $\mathbb{R}^2$ are Borel subset.

- A funct $g:\mathbb{R}^2\to\mathbb{R}$ is called a Borel function if $g^{-1}(E)\in\mathcal{B}(\mathbb{R}^2)$, for all $E\in\mathcal{B}(\mathbb{R})$

Def: Let $X, Y:\Omega\to\mathbb{R}$ be rvs. The joint distributed function of X, Y, or write F_{XY}, is defined by

$$F_{XY}:\mathbb{R}^2\to\mathbb{R}$$

$$F_{XY}(a,b) \equiv P\{X\le a,\ Y\le b\}$$

~~Remark~~:

Remark: (i) $\lim\limits_{\substack{a\to\infty\\ b\to\infty}} F_{XY}(a,b) = 1$ and

$$\lim_{\substack{a\to-\infty\\ b\to-\infty}} F_{XY}(a,b) = 0$$

From now on, all rvs are assumed to be discrete.

Write $\text{im}\, X = \{x_1, \ldots x_N\}$ $(1 \le N \le \infty)$ and

$\text{im}\, Y = \{y_1, \ldots y_M\}$ $(1 \le M \le \infty)$.

Notation:

(i) The probability function $p_X : \mathbb{R} \longrightarrow \mathbb{R}$ of X

is defined by

$$p_X(x) \equiv P\{X = x\}$$

Note: if $x \notin \text{im}\, X$, then $\{X = x\} = \phi$.

Hence $p_X(x) = 0$.

(ii). The joint prob funct $p_{XY} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

of X and Y is defined by

$$p_{XY}(x,y) = P\{X = x, Y = y\}$$

Remark ~~Note~~ (i) if $x \notin \text{im}\, X$ or $y \notin \text{im}\, Y$, then

$$p(x,y) = 0$$

Remark (ii)

$$p_X(a) = \sum_{y\in Y} p_{XY}(a,y), \quad \text{for all } a\in\mathbb{R}$$

and $$p_Y(b) = \sum_{x\in X} p_{XY}(x,b) \quad \text{for all } b\in\mathbb{R}.$$

Reason: $\sum_{y\in Y} p_{XY}(a,y)$

$$= \sum_{j=1}^{M} p_{XY}(a,y_j) = \sum_{j=1}^{M} P\{X=a, Y=y_j\}$$

$$= P\left(\bigcup_{j=1}^{M}\{X=a, Y=y_j\}\right) = P\left(\{X=a\}\cap\left(\bigcup_{j=1}^{M}\{Y=y_j\}\right)\right)$$

$$= P\{X=a\} = p_X(a).$$

(iii) If X, Y are indep, then $p_{XY}(x,y) = p_X(x)\,p_Y(y)$ $\square$

Prop: Let $g:\mathbb{R}^2 \to \mathbb{R}$ be a Borel funct.

Let $$g(X,Y): \Omega \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\omega \longmapsto (X(\omega), Y(\omega)) \longmapsto g(X(\omega), Y(\omega))$$

(Note: $g(X,Y)$ is a rv since g is Borel)

Then $$E(g(X,Y)) = \sum_{x,y} g(x,y)\, p_{XY}(x,y).$$

pf: Let $Z \equiv g(X,Y)$

Then $E(g(X,Y)) = \sum_{z} z\, P\{Z=z\}$

$$= \sum_{z} \sum_{(x,y):g(x,y)=z} z\, P\{X=x, Y=y\}$$

(since $\{Z=z\} = \bigcup_{(x,y):\, g(x,y)=z} \{X=x, Y=y\}$

and $\{X=x, Y=y\} \cap \{X=x', Y=y'\} = \phi$

if $(x,y) \neq (x',y')$.)

$$= \sum_{x,y} g(x,y)\, P\{X=x, Y=y\}$$

$\square$

Cor: If X, Y are indep, then

$$\cancel{E(g} \; E(XY) = E(X)E(Y)$$

pf: Since X, Y are indep,

$$p_{XY}(x,y) = p_X(x)\, p_Y(y)$$

Let $g(x,y) = xy$.

Then $E(XY) = E(g(X,Y)) = \sum xy\, p_X(x)\, p_Y(y) = E(X)\,E(Y)$

$\square$

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Prop: If X and Y are indep, then

$$p_{X+Y}(z) = \sum_x p_X(x)\, p_Y(z-x)$$

pf: $P\{X+Y=z\} = P\left(\bigcup_x \{X=x, Y=z-x\}\right)$

$$= \sum_x P\{X=x, Y=z-x\}$$

$$= \sum_x P\{X=x\} P\{Y=z-x\}$$

$$= \sum_x p_X(x)\, p_Y(z-x).$$

Cor: Let X, Y be Poisson rvs with parameter λ_X and λ_Y. Then $X+Y$ is a Poisson rv with parameter $\lambda_{X+Y} = \lambda_X + \lambda_Y$

pf. Aim to show: $P\{X+Y=k\} = e^{-(\lambda_X+\lambda_Y)} \dfrac{(\lambda_X+\lambda_Y)^k}{k!}$

for all $k = 0, 1, 2 \ldots$

~~Note~~ ie: $p_{X+Y}(k) = \dfrac{e^{-(\lambda_X+\lambda_Y)}}{k!} (\lambda_X+\lambda_Y)^k$

Note: $p_{X+Y}(k) = \sum_{i=0}^{\infty} p_X(i)\, p_Y(k-i)$

$$= \sum_{i=0}^{k} \frac{e^{-\lambda_X} \cdot \lambda_X^i}{i!} \cdot e^{-\lambda_Y} \frac{\lambda_Y^{k-i}}{(k-i)!} = e^{-\lambda_X-\lambda_Y} (\lambda_X+\lambda_Y)^k$$

$\square$

Def: Let X, Y be discrete r.v.s. The conditional expectation of X given that $Y=y$, write $E(X|Y=y)$, is defined by

$$E(X|Y=y) \equiv \sum_x x P\{X=x|Y=y\}$$

provided $\sum_x |x| P\{X=x|Y=y\} < \infty$.

(assume $P\{Y=y\} > 0$)

~~Remark: With the notation as above, the funct. $y \in \mathbb{R} \mapsto E(X|Y=y) \in \mathbb{R}$ is a Borel funct. Write this function~~

$$\cancel{E(X|Y) : \mathbb{R} \to \mathbb{R}}.$$

~~i.e.~~ $\cancel{E(X|Y)(y) \equiv E(X|Y=y)}$,

~~Th: E(~~

Notation: With the notation as above, define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(y) = \begin{cases} E(X \mid Y=y) & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$$

~~Write E(X|Y)~~ Then g is a Borel function

Put $E(X \mid Y) \equiv g(Y) : \Omega \to \mathbb{R}$

That is $E(X \mid Y)(\omega) = \begin{cases} E(X \mid Y=y) & \text{where } Y(\omega)=y \text{ if } E(X \mid Y=y) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$

Prop: With the notation as above

$$E(X) = E(\, E(X \mid Y))$$

pf: $E(E(X \mid Y)) = E(g(Y)) = \sum_y g(y)\, P\{Y=y\}$

$= \sum_y E(X \mid Y=y)\, P\{Y=y\} = \sum_y \sum_x x P\{X=x \mid Y=y\}\, P\{Y=y\}$

$= \sum_x x \left(\sum_y P\{X=x \mid Y=y\} P\{Y=y\} \right)$

$= \sum_x x \sum_y P\{X=x, Y=y\} = \sum_x x P\{X=x\} = E(X)$.

eg: (Geometric rv): ~~let~~ let $X : \Omega \to \{1, 2, \ldots\}$ be a ~~rv~~ discrete rv with prob function

$$p_X(k) = \begin{cases} (1-p)^{k-1} p & \text{if } k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

ie: $P\{X = k\} = (1-p)^{k-1} p$, $k = 1, 2, \ldots$

~~That is~~ $\{X = k\}$ means that in a sequence of independent trials such that the first $(k-1)$ trials are failures and the k-th trial is successful.

Find $E(X)$ and $\mathrm{var}(X)$.

Sol: Let $Y = \begin{cases} 1 & \text{if } 1^{st}\text{-trial is O.K} \\ 0 & \text{otherwise} \end{cases}$

6T

Note: $E(X) = E(E(X|Y))$

$= E(X|Y=0)\,P\{Y=0\} + E(X|Y=1)\,P\{Y=1\}$

Note: $E(X|Y=1) = \sum_{k=1}^{\infty} k\,P\{X=k|Y=1\}$

$= P\{X=1|Y=1\} = 1$

$E(X|Y=0) = E(X+1) \longrightarrow$

$$\begin{aligned} E(X|Y=0) &= \sum_{k=1}^{\infty} k\,P\{X=k|Y=0\} \\ &= \sum_{k=1}^{\infty} k\,P\{X=k-1\} \\ &= \sum_{j=0}^{\infty} (j+1)\,P\{X=j\} \\ &= \sum_{j=1}^{\infty} (j+1)\,P\{X=j\} \\ &= E(X+1) \end{aligned}$$

Then

$E(X) = E(X+1)(1-p) + 1\cdot p$

$\Rightarrow E(X) = (E(X)+1)(1-p) + p$

$\Rightarrow E(X) = \frac{1}{p}$

And $var(X) = E(X^2) - E(X)^2$

$E(X^2) = E(E(X^2|Y)) =$
$E(X^2|Y=0)\,P\{Y=0\} + E(X^2|Y=1)\,P\{Y=1\}$

Note: $E(X^2|Y=1) = 1$, and
$E(X^2|Y=0) = E[(X+1)^2] = E(X^2) + 2E(X) + 1$

$\Rightarrow E(X^2) = (E(X^2) + 2E(X) + 1)(1-p) + p$

$\Rightarrow E(X^2) = E(X^2)(1-p) + 2\cdot\frac{1}{p}(1-p) + 1-p + p$

$\Rightarrow pE(X^2) = \frac{2}{p}(1-p) + 1 \Rightarrow E(X^2) = \frac{2-p}{p^2}$

$\Rightarrow var(X) = \frac{1-p}{p^2}$ $\square$

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Def: Let X and Y be rvs. We say that X and Y are jointly continuous if there exists a Borel funct

$$f_{XY} : \mathbb{R}^2 \to [0, \infty) \quad \text{such that}$$

$$F_{XY}(a,b) = P\{X \leq a, Y \leq b\} = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{XY}(x,y)\, dx\, dy$$

for all $(a,b) \in \mathbb{R}^2$.

Remark: (i) f_{XY} is almost uniquely determined by X and Y. In fact:

$$\frac{\partial F_{XY}}{\partial x \partial y}(a,b) = f_{XY}(a,b)$$

(ii) X and Y are cont rv ~~where~~

And

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

Reason:

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$$P\{X \le a\} = P\{X \le a, Y \in \mathbb{R}\}$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{a} \left(\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \right) dx \quad , \forall a$$

$$\therefore f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy.$$

///

(iii) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1.$$

(iv) $$P\{(X,Y) \in A\} = \iint_A f_{XY}(x,y)\,dx\,dy$$

For all $A \in \mathcal{B}(\mathbb{R}^2)$.

In particular, if the size of $A = 0$, then

$$P\{(X,Y) \in A\} = 0.$$

Prop: ~~Let~~ If X and Y are jointly cont rvs with G:8

joint density f_{XY}. ~~.~~ and $g: \mathbb{R}^2 \to \mathbb{R}$ is a Borel

funct, then

$$E(g(X,Y)) = \iint_{\mathbb{R}^2} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

pf: Idea:

(i) Consider: $g = I_A$. $\quad A \in \mathcal{B}(\mathbb{R}^2)$ $\quad \cancel{A = \{(x,y) \mid x \le a,\ y \le b\}}$

(ii) Consider $g = \sum a_i I_{A_i}$

(iii) Consider $g \geq 0$, Borel funct

(iv) general case. $\square$

H.W. dead line: 21. Nov (Next Fri)

P281: 9, ~~2~~.

P246: 8.

P267, 10.

Prop: Let X and Y be jointly cnt rvs with density $f_{XY}(x,y)$. ~~Th~~ If X and Y are indep, then

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

pf: Need to show:

$$P\{X \le a, Y \le b\} = \int_{-\infty}^{a} \int_{-\infty}^{b} f_X(x) f_Y(y)\, dx\, dy$$

For all $(a,b) \in \mathbb{R}^2$.

N.B: Since ~~PX~~ X, Y indep

$$P\{X \le a, Y \le b\} = P\{X \le a\}\, P\{Y \le b\}$$

$$= \int_{-\infty}^{a} f_X(x)\, dx \int_{-\infty}^{b} f_Y(y)\, dy$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f_X(x) f_Y(y)\, dx\, dy \quad , \forall (a,b) \in \mathbb{R}^2$$

Prop: If X and Y are indep, then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \left(\equiv f_X * f_Y(z) \right)$$

pf: Need to show:

$$P\{X+Y \le a\} = \int_{-\infty}^{a} f_X * f_Y(z)\, dz, \quad \forall a \in \mathbb{R}$$

Note: Let $Z \equiv X+Y$, $A \equiv \{X+Y \le a\}$

$\Rightarrow P\{Z \le a\} = E(I_A) = E(I_{(-\infty,a]}(X+Y))$

Define : $g : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \mapsto I_{(-\infty,a]}(x+y)$

$\Rightarrow g(X,Y) = I_{\{X+Y \le a\}}$

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$$\Rightarrow P\{X+Y\le a\} = E(I_{\{X+Y\le a\}}) = E(g(X,Y))$$

$$(g: \mathbb{R}^2 \longrightarrow \mathbb{R}: I_{(-\infty,a]}(x+y)$$

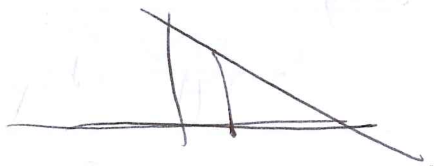
$$g(X,Y)(\omega) = I_{(-\infty,a]}(X(\omega)+Y(\omega))$$

$$= I_{\{X+Y\le a\}}(\omega)$$

$$\Rightarrow P\{X+Y\le a\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, \cancel{f_X \cdot f_Y}\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\,\cancel{(x+y)}\, I_{(-\infty,a]}(x+y)\, f_X(x) f_Y(y)\, dx\, dy$$

$$= \iint_{\{x+y\le a\}} f_X(x) f_Y(y)\, dx\, dy$$



$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a-x} f_Y(y)\, dy\right) f_X(x)\, dx$$

Let ~~z = a - y~~ Let $y = \cancel{z-x}$ $\cancel{\Rightarrow z = a - y}$ $\Rightarrow dy = dz$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a} f_Y(z-x)\, dz\right) f_X(x)\, dx$$

$$= \int_{-\infty}^{a}\left(\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx\right) dz$$

e.g: Let X and Y be indep uniform rvs over $[0,1]$

find f_{X+Y} (71)

sol: Note:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

$$= \int_0^1 f_Y(z-x)\, dx = \int_z^{z-1} f_Y(s)\,(-ds) \qquad s = z-x$$

$$= \int_{z-1}^{z} f_Y(s)\, ds$$

$$= \begin{cases} 0 & z \le 0 \\ z & 0 < z \le 1 \\ 1-(z-1) = 2-z & 1 < z \le 2 \\ 0 & z \ge 2 \end{cases}$$

Prop: If $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ are indep

then $X+Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

$\square$

pf: Later

Ch: Limit Theorems

Throughout this ~~...~~ ch, all rvs are ~~...~~ discrete.

Write $\operatorname{im} X = \{x_1, \ldots, x_N\}$ $(1 \leq N \leq \infty)$

We say that $X \geq 0$ if $x_i \geq 0$, $\forall i = 1, \ldots, N$

We say that $X \geq Y$ if $X - Y \geq 0$

Prop: (Markov's inequality): If $X \geq 0$ -

$$P\{X \geq \varepsilon\} \leq \frac{E(X)}{\varepsilon}, \quad \forall \varepsilon > 0.$$

pf: ~~Let A~~ Fix $\varepsilon > 0$. Let $A = \{X \geq \varepsilon\}$

Then $\quad I_A \leq \frac{1}{a} X$

$$\Rightarrow \underbrace{E(I_A)}_{= P(A)} \leq \frac{1}{a} E(X)$$

$\square$

Prop: (Chebyshev's inequality): Let $\mu \equiv E(X)$, $\sigma^2 \equiv \operatorname{var}(X)$

$$P\{|X - \mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}, \quad \forall \varepsilon > 0$$

pf: NB: $|X - \mu| \geq \varepsilon \Leftrightarrow (X - \mu)^2 \geq \varepsilon^2$

By Markov's inequality:

$$\underbrace{P\{(X-\mu)^2 \geq \varepsilon^2\}}_{= P\{|X - \mu| \geq \varepsilon\}} \leq \frac{E(X-\mu)^2}{\varepsilon^2} = \frac{\operatorname{var}(X)}{\varepsilon^2}$$

$\square$

Def: Let $X_1, X_2 \ldots$ be a seq of rvs

We say that $(X_1, X_2 \ldots)$ is an i.i.d seq

(independent and identically distributed)

if ①: $\{X_1, X_2 \ldots\}$ are indep

and ② $F_{X_1} = F_{X_2} = \ldots$

Remark: If $(X_1, X_2 \ldots)$ is i.i.d, then

$$E(g(X_1)) = E(g(X_2)) = \ldots \text{ for all}$$

Borel funct $g: \mathbb{R} \to \mathbb{R}$

(Reason: If X is discrete and

$\operatorname{im} X = \{x_1, x_2 \ldots x_N\}$ ($1 \leq N \leq \infty$), then

$$F_X(a) = P\{X \leq a\} = \sum_{i: x_i \leq a} P\{X = x_i\}$$

Then $P\{X = l\} = \cancel{F_X(l) - \lim_{t \to l-} F_X(t)}$

$$F_X(l) - \lim_{t \to l-} F_X(t)$$

In particular.

$E(X_1) = E(X_2) = \ldots$ and

$\operatorname{var}(X_1) = \operatorname{var}(X_2) = \ldots$

~~Th~~ Th (Weak law of large numbers):

Let $(X_1, X_2, \ldots)$ be a seq of ~~iid~~ indep rvs

Assume that $E(X_1) = E(X_2) = \cdots = \mu$, and $var(X_1) = \cdots = \cancel{var(X_n)} = \sigma^2$.

Then

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \quad \text{as } n \to \infty$$

for any $\varepsilon > 0$

pf. Let $\varepsilon > 0$. Let $\overline{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$

Then $E(\overline{X}_n) = \mu$

$$var(\overline{X}_n) = var\left(\frac{X_1 + \cdots + X_n}{n}\right)$$

$$= \frac{1}{n^2} var(X_1 + \cdots + X_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$\therefore$ By ~~Cheb~~ Chebyshev's inequality:

$$P\{|\overline{X}_n - \mu_{\overline{X}_n}| \geq \varepsilon\} \leq \frac{var(\overline{X}_n)}{\varepsilon^2}$$

$$\Rightarrow P\{|\overline{X}_n - \mu| \geq \varepsilon\} \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \quad \text{as } n \to \infty$$

$\square$

Th: (The strong law of large number)

Let $X_1, X_2 \dots$ be an i.i.d sequence. 75

Then

$$P\{ \lim_{n\to\infty} \frac{X_1+\dots+X_n}{n} = \mu \} = 1$$

pf: By considering $X_i - \mu$, we may assume

$E(X_i) = 0$, $\forall i = \dots$ since

$$\lim_n \frac{X_1+\dots+X_n}{n} = \mu \Leftrightarrow \lim_n \frac{(X_1-\mu)+\dots+(X_n-\mu)}{n} = 0$$

Let $S_n \equiv X_1 + \dots + X_n$

Note that

S_n^4 is the linear combinations of the following terms:

$$X_i^4, \quad X_i^3 X_j, \quad X_i^2 X_j^2, \quad X_i^2 X_j X_k,$$

and $X_i X_j X_k X_l$

Since (X_i) indep and $E(X_i) = 0$, $\forall i$,

$$E(X_i^3 X_j) = 0 \quad E(X_i^2 X_j X_k) = E(X_i X_j X_k X_l) = 0$$

Here

$$E(S_n^4) = \sum_{i=1}^{n} E(X_i^4) + \cancel{6\binom{n}{2}}\sum_{i\neq j} E(X_i^2 \cancel{\otimes} X_j^2)$$ 76

Since (X_i) i.i.d,

$$E(X_1^4) = E(X_2^4) = \dots \text{ and}$$

$$E(X_i^2 X_j^2) = E(X_i^2)\, E(X_j^2), \quad i \neq j$$
$$= E(X_1^2)\, E(X_1^2)$$

$$\therefore \; E(S_n^4) = n\, E(X_1^4) + 6\binom{n}{2} E(X_1^2)^2$$

$$\Rightarrow \; E(S_n^4) = n\, E(X_1^4) + 3n(n-1) E(X_1^2)^2$$

Note: $0 \leq var(X_1^2) = E(X_1^4) - E(X_1^2)^2$

$$\Rightarrow \; E(X_1^2)^2 \leq E(X_1^4)$$

$$\Rightarrow \; E(S_n^4) \leq n\, E(X_1^4) + 3n(n-1)\, E(X_1^4)$$

$$\Rightarrow \; E\left(\frac{S_n^4}{n^4}\right) \leq \frac{E(X_1^4)}{n^{3}} + \frac{3\cancel{n^2} E(X_1^4)}{n^{\cancel{4}\,2}}$$

$$\Rightarrow \; E\left(\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right) < \infty$$

$$\cancel{\Rightarrow \; P\left\{\lim_n \frac{S_n^4}{n^4} = 0\right\} = 1}$$

Note: $\sum_{n=1}^{\infty} \frac{S_n^4(\omega)}{n^4} < \infty \Rightarrow \lim_{n\to\infty} \frac{S_n^4(\omega)}{n^4} = 0$

$\Rightarrow \exists M > 0$ s.t

$\neg N$ ✓

$$E\left(\sum_{n=1}^{N} \frac{S_n^4}{n^4}\right) \leq \sum_{n=1}^{\infty} \frac{E(S_n^4)}{n^3} + \sum_{n=1}^{\infty} \frac{E(S_n^4)}{n^2} \equiv M < \infty$$

~~Let $Y_N = \sum_{n=1}^{N} \frac{S_n^4}{n^4}$ and~~

(*) ~~$P\{\lim_n Y_n = 0\} = 1$.~~ Let $Y_m \equiv \sum_{n=1}^{m} \frac{S_n^4}{n^4}$

$$Y \equiv \sum_{n=1}^{\infty} \frac{S_n^4}{n^4} : \Omega \longrightarrow [0, \infty]$$

and $Y_N \uparrow Y$

Then $E(Y) < M$.

(*) (See Next page).

$\Rightarrow$ $P\{Y = \infty\} = 0 \Rightarrow P\{Y < \infty\} = 1$

Note: $\{Y < \infty\} \subseteq \{\lim_n \frac{S_n^4}{n^4} = 0\}$

$\Rightarrow P\{\lim_n \frac{S_n^4}{n^4} = 0\} = 1$

$\Rightarrow$ $P\{\lim_n \frac{S_n}{n} = 0\} = 1$

$\square$

$\Rightarrow$ or $P\{Y_N \geq R\} \leq \frac{E(Y_N)}{R}$ (by Markov inequality) $\leq \frac{M}{R}$. $\forall R \geq 1$

Let $A_R \equiv \bigcup_N \{Y_N \geq R\} \Rightarrow P(A_R) \to 0$ as $R \to \infty$

$\vdash: P\{Y = \infty\} = 0$

On the other hand,

$P\{Y=\infty\} = P(\bigcap_{R=1}^{\infty} A_R\} = \lim_R P(A_R) = 0$

$(\because A_R \supseteq A_{R+1} \supseteq \cdots)$

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pf $\vdash$: Since $Y_m \uparrow Y$

$$\{Y = \infty\} = \bigcap_{R=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m \geq N} \{Y_m \geq R\}$$

$$= \bigcap_{R=1}^{\infty} \bigcup_{N=1}^{\infty} \{Y_N \geq R\} = \cancel{\bigcup_{N=1}^{\infty} \bigcap_{R=1}^{\infty} \{Y_N \geq R\}}$$

$\hookrightarrow$ Note $\{Y_N \geq R\} \subseteq \{Y_{N+1} \geq R\}$ as $Y_N \uparrow$

~~Note Fix N.~~ Fix R $\Rightarrow P(\bigcup_{N=1}^{\infty}\{Y_N \geq R\} = \lim_N P\{Y_N \geq R\}$

$\leq \frac{M}{R}$

$$\cancel{P\{\bigcap_{R=1}^{\infty}\{Y_N \geq R\} = \lim_{R\to\infty} P\{Y_N \geq R\}}}$$

But $E(Y_N) = \sum y P\{Y_N = y\} \leq M, \ \forall N$

$\geq \sum_{y \geq R} y P\{Y_N = y\}$

$\geq R P\{Y_N \geq R\}$

$\Rightarrow$ of $A_R \equiv \bigcup_{N=1}^{\infty}\{Y_N \geq R\}$

$\Rightarrow P(A_R) \leq \frac{M}{R}$ $\square$

$\Rightarrow P\{Y_N \geq R\} \to 0$ as $R \to \infty \Rightarrow \cancel{P\{\bigcap_{R=1}^{\infty}\{Y_N \geq R\}) = 0}$

$$\Rightarrow P\{Y = \infty\} \leq \sum_{N=1}^{\infty} P\{\bigcap_{R=1}^{\infty}\{Y_N \geq R\} = 0$$

$\square$

H.W. : ~~P327~~ P366 P12, P15

P 444, 1

eg: In a series of measurements, suppose that each measure has common mean μ and common variance $\sigma^2 = 4$. , how many measurements need to ~~maf~~ make such that 95% certain that the estimated value is accurate to ~~whi~~ within $\cancel{0.05}$ 0.5 unit. ?

sol: Let X_i be ~~th~~ i-th ~~me~~measurement.

i.e. Aim to find n s.t.

$$P\left\{ -0.5 \le \overbrace{\frac{X_1 + \cdots + X_n}{n}}^{\overline{X}_n} - \mu \le 0.5 \right\} = 0.95$$

$$\|$$

$$P\left\{ \frac{-0.5}{\frac{\sigma}{\sqrt{n}}} \le \frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \le \frac{0.5}{\frac{\sigma}{\sqrt{n}}} \right\}$$

$$\Rightarrow \Phi\left(\frac{0.5}{\frac{2}{\sqrt{n}}}\right) - \Phi\left(\frac{-0.5}{\frac{2}{\sqrt{n}}}\right) \approx 0.95 \quad (\sigma = 2)$$

$$\|$$

$$2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1$$

$n \approx 62$.

Central Limit Th:

Notation: Let $\mathcal{DF}(\mathbb{R})$ be the collection all of functions $F: \mathbb{R} \to [0,1]$ satisfying the following conds:

(i): F is increasing

(ii) $\lim_{x\to\infty} F(x)=1$ and $\lim_{x\to-\infty} F(x)=0$

(iii) F is right continuous.

Recall: For each rv X, $F_X \in \mathcal{DF}(\mathbb{R})$

~~C(F)~~ For $F \in \mathcal{DF}(\mathbb{R})$, let

Notation: • $C(F) \equiv \{x \in \mathbb{R} \mid F \text{ is cont at } x\}$

• For each $a \in \mathbb{R}$, $F(a-) = \lim_{x\to a-} F(x)$ (always exists since F is increasing and $F \leq 1$)

Hence $C(F) = \{a \in \mathbb{R} \mid F(a-) = F(a)\}$

Remark: If X is a cont rv, then

$C(F_X) = \mathbb{R}$. ie: F_X is cont on $\mathbb{R}$.

e.g: In a series of measurements, suppose that (78)

⊛ **Def:**

(i) Let (F_n) and $F \in DF(\mathbb{R})$. We say that F_n weakly conv to F, write $F_n \xrightarrow{w} F$, if

$$F_n(t) \longrightarrow F(t), \quad \forall\, t \in C(F).$$

(ii) Let (X_n) and X be r.v.s. We say that X_n conv to X weakly (or in distribution), ~~&~~ write $X_n \xrightarrow{w} X$ (or $X_n \xrightarrow{D} X$) if

$$F_{X_n} \xrightarrow{w} F_X$$

e.g: ~~Let~~ Define $F_n : \mathbb{R} \to [0,1]$ by

$$F_n(x) = \begin{cases} \frac{2}{\pi}\tan^{-1} nx & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

and

$$F(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

N.B: $C(F) = \cancel{(0,\infty)}\ \mathbb{R}\setminus\{0\}$ and $F_n(t) \longrightarrow F(t)$, $\forall t \neq 0$

$\therefore F_n \xrightarrow{w} F$. But $\cancel{F_n(t)}\ F_n(0) \longrightarrow 0$

$\therefore F_n(0) \not\longrightarrow F(0)$.

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e.g: ~~defini~~ Let X_n be a cont rv with density $f_{X_n}(t) \equiv \begin{cases} \frac{2}{\pi} \cdot \frac{n}{1+n^2 t^2} & t \geq 0 \\ 0 & t < 0 \end{cases}$

$\Rightarrow$ ~~P~~ $F_{X_n}(x) = P\{X_n \leq x\} = \int_{0}^{x} f_n(t)\, dt = \frac{2}{\pi} \cdot \tan^{-1} nx$ if $x \geq 0$

$\therefore F_n(x) = \begin{cases} \frac{2}{\pi} \tan^{-1} nx & x \geq 0 \\ 0 & x < 0 \end{cases}$

~~Define~~ Let $X : \Omega \to \mathbb{R}$ be a r.v such that $X(\Omega) \equiv 0$.

Then $F_X(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

So $X_n \xrightarrow{\mathcal{D}} X$

Note: X is a discrete r.v

(CLT): Let (X_n) be a seq of i.i.d r.vs

Let $\overline{X}_n \equiv \frac{X_1 + \cdots + X_n}{n}$. $\mu \equiv E(X_i)$, $\sigma^2 = var(X_i)$

Then $\frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{D} Z \sim N(0,1)$

· Let X be a r.v.

Def: The characteristic function (CF) of a rv X is defined by the map

$$\varphi_X : \mathbb{R} \longrightarrow \mathbb{C}.$$

$$\varphi_X(\theta) \equiv E(e^{i\theta X}) = E(\cos \theta X) + i\, E(\sin \theta X).$$

Remark:

①: If X is discrete, then

$$E(e^{i\theta X}) = \sum_{k=1}^{N} e^{i\theta x_k} P\{X = x_k\}$$

②: If X is cont with density f_X

$$E(e^{i\theta X}) = \int_{\mathbb{R}} e^{i\theta x} f_X(x)\, dx$$

THEN: $E(e^{i\theta X})$ must exist if $E(X)$ exists.

Prop: With the notation as above:

(i): $|\varphi_X(\theta)| \leq 1, \quad \forall \theta \in \mathbb{R}$

(ii) $\varphi_{-X}(\theta) = \overline{\varphi_X(\theta)}$

(iii) ~~$\varphi_X \in C^2(\mathbb{R})$ and $\varphi_X''(\theta)$~~

$\varphi_X'(\theta) = E(iX e^{i\theta X})$, $\varphi_X''(\theta) = E((iX)^2 e^{i\theta X})$
$= -E(X^2 e^{i\theta X})$

& both exists if $E(X)$ and $var(X)$ exist.

(iv) $\varphi_{\alpha X+\beta}(\theta) = e^{i\beta\theta} \varphi_X(\alpha X)$

(v): if X and Y are indep, then

$$\varphi_{X+Y}(\theta) = \varphi_X(\theta)\varphi_Y(\theta).$$

|||

e.g: ~~Suppo~~ If $P\{X=1\} = \frac{1}{2} = P\{X=-1\}$, then

~~φ~~ $\varphi_X(\theta) = \cos\theta$

Sol: $\varphi_X(\theta) = E(e^{i\theta X}) = e^{i\theta \cdot 1} P\{X=1\} + e^{i\theta(-1)} P\{X=-1\}$

$$= \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \cos\theta$$

□

e.g.: ① If $X \sim N(0,1)$, then

$$E(e^{i\theta X}) = e^{-\frac{\theta^2}{2}}$$

② : If $X \sim N(\mu,\sigma^2)$, then

$$\varphi_X(\theta) = e^{i\mu\theta - \frac{1}{2}\sigma^2\theta^2}$$

pf ①: if $X \sim N(0,1)$, then

$$E(e^{i\theta X}) = \int_{-\infty}^{\infty} e^{i\theta x} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x^2-2i\theta x)}{2}}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-i\theta)^2+\theta^2}{2}}\, dx$$

$$= e^{-\frac{1}{2}\theta^2}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-i\theta)^2}{2}}}{\sqrt{2\pi}}\, dx = e^{-\frac{1}{2}\theta^2}$$

②: Note: if $X \sim N(\mu,\theta) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$

$$\Rightarrow \varphi_{\frac{1}{\sigma}X - \frac{\mu}{\sigma}}(\theta) = e^{-\frac{1}{2}\theta^2}$$

$$\parallel$$

$$e^{i\frac{-\mu}{\sigma}}\varphi_X\left(\frac{1}{\sigma}\theta\right)$$

Put $t = \frac{1}{\sigma}\theta \Rightarrow \varphi_X(t) = e^{i\frac{\mu}{\sigma}\theta} \cdot e^{-\frac{1}{2}\sigma^2 t^2}$

$$= e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

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$\square$

Th (Levy's Inversion formula): Let φ_X be the CF of a rv X. For $a < b$, then

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ixa} - e^{-ixb}}{ix} \varphi_X(x)\, dx \quad \cdots$$

$$\overset{(*)}{=} \frac{1}{2}(F_X(b) + F_X(b-)) - \frac{1}{2}(F_X(a) + F_X(a-))$$

In particular, if X is cont, then

$$(*) = F_X(b) - F_X(a)$$

$\square$

Remark$^{(*)}$: If $\varphi_X = \varphi_Y$, then

$$F_X(b) = F_Y(b), \quad \forall b \in C(F_X) \cap C(F_Y)$$

In particular, if X and Y are cont, then

$$F_X(b) = F_Y(b), \quad \forall b \in \mathbb{R}$$

$\square$

(87)

e.g.: Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$,

If X and Y are indep, then

$$X+Y \sim N(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$$

pf: Need to show

$$\varphi_{X+Y}(\theta) = e^{i(\mu_X+\mu_Y)\theta - \frac{1}{2}(\sigma_X^2+\sigma_Y^2)\theta^2}$$

In fact: $\varphi_{X+Y}(\theta) = \varphi_X(\theta)\,\varphi_Y(\theta)$

$$= e^{i\mu_X\theta - \frac{1}{2}\sigma_X^2\theta^2} \cdot e^{i\mu_Y\theta - \frac{1}{2}\sigma_Y^2\theta^2}$$

$\square$

AGA

Th: (Levi's ~~continuous~~ continuity th): Let (X_n) be a seq of ~~r.v.s.~~ ~~d.f.s~~ rvs.

Suppose that $g(\theta) \stackrel{def}{\equiv} \lim_n \varphi_{X_n}(\theta)$ exists $\forall \theta \in \mathbb{R}$

and $g(\cdot)$ is cont at $\theta = 0$.

THEN: $\exists$ $(\Omega, \mathcal{E}, \mathbb{P})$ and $\exists$ rv $X : \Omega \to \mathbb{R}$ st

$$X_n \xrightarrow{\mathcal{D}} X \quad \text{and} \quad \varphi_X = g$$

~~Th~~ (Proof CLT):

Let $Z_n \equiv \dfrac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$, $Z \sim N(0,1)$

Need to show that $\lim_{n\to\infty} \varphi_{Z_n}(\theta) \stackrel{*}{=} e^{-\frac{1}{2}\theta^2}$, $\forall \theta \in \mathbb{R}$

(Reason: By Levi's continuity, $\exists$ rv Z st

$$Z_n \xrightarrow{\mathcal{D}} Z \quad \text{and} \quad \varphi_Z = e^{-\frac{1}{2}\theta^2}$$

~~By Levi's~~ Since if $X \sim N(0,1)$, then

$$\varphi_X = e^{-\frac{1}{2}\theta^2} \Rightarrow \varphi_X = \varphi_Z$$

$\therefore$ by inversion formula, $F_X = F_Z$

ie: $Z \sim N(0,1)$.

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Now pf $\circledast$: $\lim_{n\to\infty} \varphi_{Z_n}(\theta) = e^{-\frac{1}{2}\theta^2}$

By considering $X_i - \mu$, we may assume

$\mu = 0$.

Then $Z_n = \dfrac{X_1 + \cdots + X_n}{\sqrt{n}\,\sigma} = \dfrac{X_1}{\sqrt{n}\,\sigma} + \cdots + \dfrac{X_n}{\sqrt{n}\,\sigma}$

Note: $\left(\dfrac{X_1}{\sqrt{n}\,\sigma}, \ldots, \dfrac{X_n}{\sqrt{n}\,\sigma}\right)$ is i.i.d

$$\Rightarrow \varphi_{Z_n}(\theta) = \varphi_{\frac{X_1}{\sqrt{n}\sigma}}(\theta) \cdots \varphi_{\frac{X_1}{\sqrt{n}\sigma}}(\theta) = \left[\varphi_{\frac{X_1}{\sqrt{n}\sigma}}(\theta)\right]^n = \left[\varphi_{X_1}\left(\frac{\theta}{\sqrt{n}\,\sigma}\right)\right]^n$$

Note: By Taylor th:

$$\varphi_{X_1}(t) = \varphi_{X_1}(0) + \varphi'_{X_1}(0)\,t + \frac{\varphi''_{X_1}(0)}{2!}t^2 + o(t^2)$$

where $\dfrac{o(t^2)}{t^2} \to 0$ as $t \to 0$

Note: $\varphi_{X_1}(0) = 1$.

$$\varphi'_{X_1}(0) = E(iX_1 e^{itX_1})\Big|_{t=0} = iE(X) = 0$$

$$\varphi''_{X_1}(0) = E((iX_1)^2 e^{itX_1})\Big|_{t=0} = -E(X_1^2) = -\sigma^2$$

$(\because E(X_1) = 0)$

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$$\therefore \varphi_{X_1}\left(\frac{\theta}{\sqrt{n}\sigma}\right) = 1 + \frac{1}{2}(-\sigma^2)\cdot\frac{\theta^2}{n\sigma^2} \cancel{\sigma} + o\left(\frac{\theta^2}{n\sigma^2}\right)$$

$$= 1 - \frac{1}{2}\theta^2 + o\left(\frac{\theta^2}{n\sigma^2}\right)$$

$$\Rightarrow \varphi_{Z_n}(\theta) = \lim_n \left[1 - \frac{1}{2}\frac{\theta^2}{n} + o\left(\frac{\theta^2}{n\sigma^2}\right)\right]^n$$

$$= \lim_n \left(1 - \frac{1}{2}\frac{\theta^2}{n}\right)^n = e^{-\frac{1}{2}\theta^2}$$

$\square$

H.W. P509. 14, 20